

# N=4 Super NLS-mKdV Hierarchies

E. Ivanov<sup>a,1</sup>, S. Krivonos<sup>a,2</sup> and F. Toppan<sup>b,3</sup>

<sup>(a)</sup> *JINR-Bogoliubov Laboratory of Theoretical Physics,  
141980 Dubna, Moscow Region, Russia*

<sup>(b)</sup> *Dipartimento di Fisica, Università di Padova,  
Via Marzolo 8, 35131 Padova, Italy*

## Abstract

$N = 2$  extension of affine algebra  $sl(2) \widehat{\oplus} u(1)$  possesses a hidden global  $N = 4$  supersymmetry and provides a second hamiltonian structure for a new  $N = 4$  supersymmetric integrable hierarchy defined on  $N = 2$  affine supercurrents. This system is an  $N = 4$  extension of at once two hierarchies,  $N = 2$  NLS and  $N = 2$  mKdV ones. It is related to  $N = 4$  KdV hierarchy via a generalized Sugawara-Feigin-Fuks construction which relates  $N = 2$   $sl(2) \widehat{\oplus} u(1)$  algebra to “small”  $N = 4$  SCA. We also find the underlying affine hierarchy for another integrable system with the  $N = 4$  SCA second hamiltonian structure, “quasi”  $N = 4$  KdV hierarchy. It respects only  $N = 2$  supersymmetry. For both new hierarchies we construct scalar Lax formulations. We speculate that any  $N = 2$  affine algebra admitting a quaternionic structure possesses  $N = 4$  supersymmetry and so can be used to produce  $N = 4$  supersymmetric hierarchies. This suggests a way of classifying all such hierarchies.

March 1997  
JINR E2-97-108  
DFPD 97-TH-11  
hep-th/9703224

*E-Mail:*

1) *eivanov@thsun1.jinr.dubna.su*

2) *krivonos@thsun1.jinr.dubna.su*

3) *toppan@mvrpd5.pd.infn.it*

**1. Introduction.** The integrable hierarchies of differential equations in  $1 + 1$  dimensions have been widely studied in the last several years both in the physical and in the mathematical contexts. There is an ever-growing evidence that their relevance should not be confined to the realm of pure mathematics, but rather their beautiful mathematical structures show themselves naturally when investigating physical problems. At first they appeared in connection with the discretized versions, via the matrix-models approach, of 2-dimensional gravity (see [1] and references therein). Recently they sprang up rather unexpectedly in the domain of 4-dimensional field theories as an underlying structure of the  $N = 2$  super Yang-Mills theories in the Seiberg-Witten approach [2]. The remarkable relation of the KdV-type hierarchies, via their second hamiltonian structure, to the conformal algebras (Virasoro algebra and its extensions) establishes a link between such hierarchies and the string theories which has still to be fully appreciated.

One is naturally led to investigate supersymmetric extensions of the above bosonic hierarchies. Besides  $N = 1$  supersymmetric systems which were studied in a lot of papers starting from refs. [3], it is very interesting to consider the larger  $N$  cases. One of the main sources of interest in extended supersymmetric hierarchies with  $N > 1$  is that they could lead to a sort of “unification or grand-unification” of known hierarchies. It could happen that seemingly unrelated bosonic or lower supersymmetric ( $N = 1, 2$ ) hierarchies are different manifestations of a single “unifying” larger  $N$  supersymmetric hierarchy.

During the last years,  $N = 2$  supersymmetric hierarchies were a subject of many exhaustive studies. Much less was known about systems with higher  $N$ . In a series of papers [4-7] the first example of hierarchy with  $N = 4$  supersymmetry,  $N = 4$  super KdV system, was discovered and studied. It is related, through its second hamiltonian structure, to “small”  $N = 4$  superconformal algebra, is bi-hamiltonian and admits a Lax-pair formulation. It encompasses two different  $N = 2$  KdV hierarchies, the  $a = 4$  and  $a = -2$  ones, which follow from it via appropriate reductions. Recently, one more integrable hierarchy with “small”  $N = 4$  SCA as the second hamiltonian structure has been found [8]. As distinct from the “genuine”  $N = 4$  KdV system,  $N = 4$  global supersymmetry is broken to  $N = 2$  in this new “quasi”  $N = 4$  KdV hierarchy. It yields, by proper reductions, some previously unknown KdV type systems with lower supersymmetry.

In this paper we extend the class of  $N = 4$  supersymmetric hierarchies by suggesting a general Lie-algebraic framework for them.

Our consideration relies upon the well-known property that generalized KdV and KdV type hierarchies associated with (super)conformal algebras and  $W$  algebras as the hamiltonian structure can be induced, via a Miura - type transformation, from the hierarchies associated in a similar way with appropriate (super)affine algebras. Thus the latter hierarchies underly the former ones, and are in a sense more fundamental. In the algebraic language, such a correspondence amounts to the existence of the Sugawara-Feigin-Fuks (SFF) type (or coset) construction of the former (super)algebras in terms of the latter ones. For  $N = 2$  hierarchies a few examples of this sort were explicitly elaborated. These are, e.g., the relationships between  $N = 2$  KdV and  $N = 2$  Boussinesq hierarchies, on one hand, and their affine counterparts, on the other [8, 9]. The relevant Miura transformations relate the corresponding second hamiltonian structures,  $N = 2$  SCA and  $N = 2$   $W_3$ , to  $N = 2$  extensions of the affine algebra  $u(1) \widehat{\oplus} u(1)$  and of two copies of the latter, respectively.

Our proposal is to construct  $N = 4$  supersymmetric hierarchies in terms of  $N = 2$  extended

affine algebras possessing a hidden  $N = 4$  structure. This class embraces the algebras admitting a quaternionic structure. Their local bosonic parts were listed in [10] while analyzing the question as to which group-manifold WZNW sigma models admit  $N = 4$  supersymmetric extension. The simplest non-trivial  $N = 2$  affine algebra of this kind is  $N = 2 \widehat{sl(2) \oplus u(1)}$ , and this is the case we treat in detail in the present paper.

We start with the formulation of  $N = 2 \widehat{sl(2) \oplus u(1)}$  in terms of two pairs of spin  $1/2$   $N = 2$  supercurrents subjected to the chirality-type constraints. We show that both the structure relations of the algebra and constraints are closed under some hidden nonlinear  $N = 2$  supersymmetry transformations. Together with the manifest  $N = 2$  supersymmetry, they constitute  $N = 4$  supersymmetry. The affine supercurrents form an irreducible multiplet of this  $N = 4$  SUSY. Then we demonstrate the existence of an infinite set of  $N = 4$  invariant quantities in involution constructed out of the affine supercurrents. This new hierarchy with  $N = 2 \widehat{sl(2) \oplus u(1)}$  as the second hamiltonian structure can be assigned the name  $N = 4$  super NLS-mKdV hierarchy because it leads to  $N = 2$  NLS hierarchy of ref. [11, 12] and to  $N = 2$  mKdV hierarchy as its two non-equivalent reductions. Further, we set up, via a generalized SFF construction, three spin  $1$   $N = 2$  supercurrents of the “small”  $N = 4$  SCA. We demonstrate that in terms of the  $N = 4$  SCA supercurrents the  $N = 4$  NLS-mKdV conserved quantities coincide with those of  $N = 4$  KdV hierarchy. Thus  $N = 4$  NLS-mKdV and KdV hierarchies are related to each other in the way resembling the relation between ordinary mKdV and KdV ones. An essential difference is, however, that the affine algebra relevant to our case is the non-abelian  $N = 2 \widehat{sl(2) \oplus u(1)}$  algebra. The “quasi”  $N = 4$  KdV hierarchy of ref. [8] is also contained in the enveloping algebra of  $N = 2 \widehat{sl(2) \oplus u(1)}$ . It corresponds to another choice of the evolution equations for the affine supercurrents, with the hamiltonians breaking global  $N = 4$  supersymmetry down to  $N = 2$ .

**2. Hidden  $N = 4$  supersymmetry of  $N = 2 \widehat{sl(2) \oplus u(1)}$  algebra.** The  $N = 2$  affine  $\widehat{sl(2) \oplus u(1)}$  algebra is constituted by the two pairs of  $N = 2$  spin  $1/2$  fermionic superfields  $F(Z)$ ,  $\bar{F}(Z)$  and  $H(Z)$ ,  $\bar{H}(Z)$ ,  $Z = (x, \theta, \bar{\theta})$  being coordinates of  $N = 2$  superspace. These supercurrents satisfy the following classical Poisson Bracket (PB) relations [13]

$$\{H(1), \bar{H}(2)\} = D\bar{D}\delta(1, 2) \quad (1)$$

$$\begin{aligned} \{H(1), F(2)\} &= DF\delta(1, 2), \quad \{H(1), \bar{F}(2)\} = -D\bar{F}\delta(1, 2), \\ \{\bar{H}(1), F(2)\} &= -\bar{D}F\delta(1, 2), \quad \{\bar{H}(1), \bar{F}(2)\} = \bar{D}\bar{F}\delta(1, 2), \end{aligned} \quad (2)$$

$$\{F(1), \bar{F}(2)\} = [(D + H)(\bar{D} + \bar{H}) + F\bar{F}]\delta(1, 2), \quad (3)$$

all other PBs vanishing. In these relations,  $1, 2 \equiv Z_{1,2}$ ,  $D, \bar{D}$  are  $N = 2$  spinor derivatives

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2}\bar{\theta}\partial_x, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2}\theta\partial_x, \quad D^2 = \bar{D}^2 = 0, \quad \{D, \bar{D}\} = -\partial_x, \quad (4)$$

$\delta(1, 2)$  is the  $N = 2$  superspace delta-function

$$\delta(1, 2) = \delta(x_1 - x_2)(\theta_1 - \theta_2)(\bar{\theta}_1 - \bar{\theta}_2),$$

and the differential operators in the r.h.s. are evaluated at the point  $Z_1$ . The derivatives in the r.h.s. of these relations are assumed to act freely to the right.

The PB set (1) - (3) is closed, i.e. the Jacobi identities are valid, only provided the involved supercurrents are subjected to the constraints [13]

$$DH = 0, \quad \overline{D}\overline{H} = 0, \quad (5)$$

$$(D + H)F = 0, \quad (\overline{D} - \overline{H})\overline{F} = 0. \quad (6)$$

Thus  $H$  and  $\overline{H}$  are ordinary chiral and anti-chiral superfields while  $F$  and  $\overline{F}$  are “covariantly” chiral and anti-chiral ones. It is easy to check that constraints (5) - (6) are consistent with the above PB relations, in the sense that the PBs between them and with all supercurrents are vanishing on the shell of constraints. Note that the non-linear term in the r.h.s. of (3) is “fake”: at the component (and even at the  $N = 1$  superfield) level there is no non-linearity in the algebra. Its presence in (3) is the price for manifest  $N = 2$  supersymmetry.

As is seen from (1), the superfields  $H$  and  $\overline{H}$  generate the Cartan  $N = 2$   $u(1) \widehat{\oplus} u(1)$  subalgebra of the  $N = 2$   $sl(2) \widehat{\oplus} u(1)$ . The relations (2) tell us about the properties of  $F$  and  $\overline{F}$  with respect to two independent  $u(1)$  charges inherent in  $N = 2$   $sl(2) \widehat{\oplus} u(1)$ . It is assumed that  $H$  and  $\overline{H}$  have the same transformation properties under the global  $U(1)$  automorphism group of  $N = 2$ ,  $1D$  superalgebra as the derivatives  $D$  and  $\overline{D}$ .

We note that for classical affine algebras the central charge corresponding to the Kac-Moody level can be rescaled at will (using the freedom to rescale the fields as well as the Poisson brackets). Our convention is dictated by further convenience.

The algebra  $N = 2$   $sl(2) \widehat{\oplus} u(1)$  is a particular representative of a wide class of  $N = 2$  affine algebras. Actually,  $N = 2$  superextension can be defined for any affine algebra (and superalgebra) admitting a complex structure [13], [14].  $N = 2$  supercovariance of these extensions is manifest when they are formulated in terms of (covariantly) chiral and anti-chiral  $N = 2$  superfields. Nevertheless, it could be equally revealed in the  $N = 1$  superfield formulation. In such a formulation,  $N = 2$  supercurrents are represented by pairs of real spin  $1/2$   $N = 1$  superfields, and rigid  $N = 2$  supersymmetry is realized as some transformations mixing  $N = 1$  superfields inside each pair. Then the statement that the given  $N = 1$  extended affine algebra is actually  $N = 2$  extended amounts to the covariance of the defining  $N = 1$  superfield PBs under these transformations.

It is natural to ask whether the variety of  $N = 2$  affine (super)algebras contains as a subclass the algebras which reveal covariance under some extra supersymmetries. As should be clear from the reasoning just given, one way to answer this question is to explicitly construct the transformations of these additional supersymmetries on the affine  $N = 2$  supercurrents and to check covariance of the relevant PBs and chirality constraints with respect to them.

Let us show that the algebra  $N = 2$   $sl(2) \widehat{\oplus} u(1)$  possesses a hidden  $N = 2$  supersymmetry which, together with the manifest  $N = 2$  supersymmetry, form  $N = 4$  supersymmetry.

This extra supersymmetry is realized by the following non-linear transformations

$$\begin{aligned} \delta H &= \epsilon D\overline{F} + \overline{\epsilon} H F, & \delta \overline{H} &= \overline{\epsilon} \overline{D} F - \epsilon \overline{H} \overline{F} \\ \delta F &= -\epsilon D\overline{H} - \epsilon (H\overline{H} + F\overline{F}), & \delta \overline{F} &= -\overline{\epsilon} \overline{D} H - \overline{\epsilon} (H\overline{H} + F\overline{F}), \end{aligned} \quad (7)$$

$\epsilon, \overline{\epsilon}$  being the corresponding infinitesimal parameters. It can be easily checked that the above transformations indeed realize an  $N = 2$  supersymmetry, i.e. that their commutators close to give, for any superfield  $\Psi \equiv \{H, \overline{H}, F, \overline{F}\}$

$$[\delta_1, \delta_2]\Psi = (\overline{\epsilon}_1 \epsilon_2 - \overline{\epsilon}_2 \epsilon_1) \partial_x \Psi. \quad (8)$$

The hidden and manifest  $N = 2$  supersymmetry transformations commute with each other and so form an  $N = 4$  supersymmetry. The two pairs of affine  $N = 2$  supercurrents  $F, \overline{F}$  and  $H, \overline{H}$  are unified into an irreducible  $N = 4$  supermultiplet.

A remarkable property of the above transformations is that they preserve the (nonlinear) chirality and antichirality constraints (5), (6).

Moreover, it is a matter of tedious though straightforward computation to check that the defining relations of the  $N = 2$   $\widehat{sl(2) \oplus u(1)}$  algebra, eqs. (1) - (3) (together with the vanishing PBs), are closed under the transformations (7). One varies both sides of these relations and finds the variations being the same.

Thus the algebra  $N = 2$   $\widehat{sl(2) \oplus u(1)}$  reveals a hidden  $N = 4$  supersymmetry and hence it is in fact a minimal  $N = 4$  extension of  $\widehat{sl(2) \oplus u(1)}$ . We expect that this  $N = 4$  covariance can be made manifest by combining the affine  $N = 2$  supercurrents into a single  $N = 4$  superfield, e.g., in the framework of  $N = 4$  harmonic superspace [4].

It is reasonable to assume that there exists the whole class of affine algebras and superalgebras allowing for an  $N = 4$  extension along similar lines. A plausible conjecture is that these algebras are those which admit a quaternionic structure. The full list of such algebras and corresponding groups was given in [10]. Though we are not still aware of the general proof of this conjecture, we have checked it on more examples. One of them is the  $N = 2$  affine  $\widehat{sl(3)}$  algebra. Its basic relations also reveal covariance under the appropriate  $N = 4$  transformations, we are going to consider this case in detail in a forthcoming publication. Another, simplest example is an  $N = 2$  extension of the affine algebra  $\oplus_{i=1}^4 \widehat{u(1)_i}$ . It is constituted by the two conjugated mutually commuting pairs of chiral and anti-chiral  $N = 2$  superfields  $H_\alpha, \overline{H}_\alpha$ , ( $\alpha = 1, 2$ ),

$$\begin{aligned} \{H_\alpha, H_\beta\} &= 0, \quad \{H_\alpha, \overline{H}_\beta\} = \delta_{\alpha\beta} D\overline{D}\delta(1, 2), \\ DH_\alpha &= \overline{D}\overline{H}_\alpha = 0. \end{aligned} \quad (9)$$

It is straightforward to check the covariance of (9) under the linearized version of (7)

$$\delta H_1 = \epsilon D\overline{H}_2, \quad \delta \overline{H} = \overline{\epsilon} \overline{D}H_2, \quad \delta H_2 = -\epsilon D\overline{H}_1, \quad \delta \overline{H}_2 = -\overline{\epsilon} \overline{D}H_1. \quad (10)$$

These transformations has the same Lie bracket structure as (7). Later on we will argue that this simplest  $N = 2$  affine algebra with  $N = 4$  structure bears no direct relation to  $N = 4$  KdV hierarchy of refs. [4-6], in contrast to  $N = 2$   $\widehat{sl(2) \times u(1)}$  which does.

**3.  $N = 4$  invariant hamiltonians and flows.** In the previous section we have shown that the algebra  $N = 2$   $\widehat{sl(2) \oplus u(1)}$  carries an  $N = 4$  structure and is in fact an  $N = 4$  extension of  $\widehat{sl(2) \oplus u(1)}$ . We may wonder whether it generates, as a second hamiltonian structure, some  $N = 4$  supersymmetric hierarchy of evolution equations for  $F, \overline{F}, H, \overline{H}$ , or, in other words, whether its enveloping algebra contains an infinite set of  $N = 4$  supersymmetric hamiltonians in involution. The answer is positive, and now we explicitly furnish first three such  $N = 4$  invariant hamiltonians. The proof that we indeed face an integrable hierarchy will be given later by presenting the scalar Lax operator for it.

The hamiltonian densities  $\mathcal{H}_i$  given below are invariant under (7) up to total derivatives and correspond to integral spin dimension  $i = 1, 2, 3$ , respectively. In addition, we require them to be globally chargeless ( $Q(\mathcal{H}_i) = 0$ ) with respect to a charge operator  $Q$  such that  $Q(H) = Q(\overline{H}) = 0$ ,  $Q(F) = 1$ ,  $Q(\overline{F}) = -1$ . The reason why we impose this condition is the desire to

reproduce the conserved quantities of  $N = 2$  NLS hierarchy in the limit  $H = \bar{H} = 0$ ; requiring this invariance amounts to the property that the reduced hamiltonians “live” on the quotient of  $N = 2$   $\widehat{sl(2) \oplus u(1)}$  over its Cartan subalgebra  $u(1) \oplus u(1)$ , which is the characteristic feature of  $N = 2$  NLS hierarchy [12]. For sure, this choice does not yield the most general set of  $N = 4$  invariant hamiltonians one can construct (and does not even correspond to the most general hierarchy, see however the remarks in section 4), but it is the only choice which allows us to recover  $N = 2$  NLS hierarchy.

It is of interest to point out that  $\mathcal{H}_2$  is uniquely defined by the requirements of  $Q(\mathcal{H}_2) = 0$  and  $N = 4$  invariance, i.e. it is of no need to resort to the involutivity reasonings to specify its coefficients. We have

$$\begin{aligned}
\mathcal{H}_1 &= F\bar{F} + H\bar{H} \\
\mathcal{H}_2 &= F'\bar{F} - H'\bar{H} - (D\bar{H} + \bar{D}H)(H\bar{H} + F\bar{F}) - 2H\bar{H}F\bar{F} \\
\mathcal{H}_3 &= F''\bar{F} + H''\bar{H} - 3\bar{D}H F\bar{F}' - 6\bar{D}H F'\bar{F} + 3\bar{D}H H'\bar{H} + 6\bar{D}H F\bar{F}' - H\bar{D}F\bar{F}' \\
&\quad - 5H'\bar{D}F\bar{F} + \frac{3}{2}H'D\bar{H}\bar{H} - 4\bar{H}F'D\bar{F} - \bar{H}'F\bar{D}\bar{F} - \bar{D}F\bar{F}D\bar{F} \\
&\quad + 9\bar{D}H\bar{D}H F\bar{F} + \bar{D}H\bar{D}H H'\bar{H} - 8\bar{D}H H\bar{D}F\bar{F} + 3\bar{D}H H\bar{D}\bar{H}\bar{H} \\
&\quad + 9\bar{D}H\bar{D}\bar{H}F\bar{F} + 8\bar{D}H\bar{H}F\bar{D}\bar{F} + H\bar{D}H\bar{D}\bar{H}\bar{H} + 2H\bar{H}\bar{D}F\bar{D}\bar{F}
\end{aligned} \tag{11}$$

(hereafter, the spatial derivative is denoted with a prime).

The corresponding first and second flows are given by

$$\frac{\partial \Psi}{\partial t_1} = \Psi' \tag{12}$$

for any  $\Psi \equiv \{H, \bar{H}, F, \bar{F}\}$ , and

$$\begin{aligned}
\frac{\partial F}{\partial t_2} &= -F'' + 2\bar{D}H F' + 2\bar{D}H'F + 2\bar{D}\bar{H}F' + 2\bar{D}F\bar{F}D\bar{F} - 2FF'\bar{F} \\
&\quad + 2HD\bar{H}\bar{D}F + 2H\bar{H}F' + 2H\bar{H}'F - 2\bar{D}H H'\bar{H}F + 4H\bar{D}F\bar{F}' , \\
\frac{\partial H}{\partial t_2} &= H'' + 2\bar{D}H H' + 2HD\bar{H}' + 2H'D\bar{H} + 2\bar{D}H F\bar{D}\bar{F} - 2H\bar{D}F\bar{D}\bar{F} \\
&\quad + 2HF'\bar{F} + 2H'F\bar{F} - 2HD\bar{H}F\bar{F} - 2H\bar{H}F\bar{D}\bar{F} .
\end{aligned} \tag{13}$$

We do not present the third flow equations in view of their complexity.

**4. Generalized Sugawara construction and relation to  $N = 4$  KdV.** To clarify the meaning of the above involutive sequence of  $N = 4$  supersymmetric hamiltonians, let us apply to the Sugawara type construction on the algebra  $N = 2$ ,  $\widehat{sl(2) \oplus u(1)}$ . We will show that it naturally leads to the classical version of “small”  $N = 4$  SCA with the arbitrary central charge (proportional to that of the affine algebra). In terms of composite  $N = 4$  SCA supercurrents the hierarchy presented in the previous section is recognized as  $N = 4$  KdV hierarchy.

We start with the standard classical  $N = 2$  SFF stress-tensor (with a fixed central charge, see remark in Sect. 2)

$$J = H\bar{H} + F\bar{F} + D\bar{H} + \bar{D}H . \tag{14}$$

As a consequence of the  $N = 2$  affine PBs (1) - (3) it satisfies the PB of  $N = 2$  SCA

$$\{J(1), J(2)\} = \left( J\partial + \partial J + DJ\overline{D} + \overline{D}JD + \partial[D, \overline{D}] \right) \delta(1, 2) , \quad (15)$$

where as before the differential operator in the r.h.s. is evaluated at  $Z_1$  and all derivatives act freely. Considering the transformation properties of  $J$  under the transformations (7), we find that it forms an irreducible *linear*  $N = 4$  supermultiplet together with the spin 1 chiral and anti-chiral supercurrents

$$\Phi \equiv D\overline{F} , \quad \overline{\Phi} \equiv \overline{D}F , \quad D\Phi = \overline{D}\overline{\Phi} = 0 . \quad (16)$$

$$\delta J = -\epsilon\overline{D}\Phi - \bar{\epsilon}D\overline{\Phi} , \quad \delta\Phi = \bar{\epsilon}DJ , \quad \delta\overline{\Phi} = \epsilon\overline{D}J . \quad (17)$$

It is straightforward to check that the original PBs (1) - (3) imply the following non-vanishing PBs for  $\Phi, \overline{\Phi}$

$$\begin{aligned} \{J(1), \Phi(2)\} &= -\left( \Phi\overline{D}D + \overline{D}\Phi D \right) \delta(1, 2) , \quad \{J(1), \overline{\Phi}(2)\} = -\left( \overline{\Phi}D\overline{D} + D\overline{\Phi}\overline{D} \right) \delta(1, 2) , \\ \{\Phi(1), \overline{\Phi}(2)\} &= \left( \partial D\overline{D} + D\overline{D}\partial \right) \delta(1, 2) . \end{aligned} \quad (18)$$

Together with (15) they form a closed set of PBs which defines a classical version of “small”  $N = 4$  SCA [5,8] (it is closed with respect to both Jacobi identities and rigid  $N = 4$  transformations (17))<sup>1</sup>.

As was remarked previously, in the present case it is not essential which specific value is ascribed to the central charge, since for classical algebras it can be rescaled to any value through a rescaling of the fields and PBs. However, it is worth mentioning that the central charge of the above  $N = 4$  SCA is strictly related to that of the underlying  $N = 2$   $\widehat{sl(2) \oplus u(1)}$  algebra. In particular, with our convention on the central charge in the structure relations of  $N = 2$   $\widehat{sl(2) \oplus u(1)}$  the coefficient before the Feigin-Fuks term in (14) should be strictly 1 in order the PBs of  $N = 4$  SCA were closed. We also note that now the rigid supersymmetry (7) can be recovered as a part of local supersymmetry generated by  $\Phi, \overline{\Phi}$  on the affine  $N = 2$  supercurrents through the PBs (1) - (3).

The  $N = 4$  SFF realization (14), (16) is the key which allows us to establish a connection between our  $N = 4$  NLS-mKdV hierarchy and the  $N = 4$  KdV. It is a simple exercise to check that the first three hamiltonian densities in (11), up to full derivatives, are expressed in terms of the above composite  $N = 4$  supercurrents

$$\mathcal{H}_1 = J , \quad \mathcal{H}_2 = -\frac{1}{2} \left( J^2 - 2\Phi\overline{\Phi} \right) , \quad \mathcal{H}_3 = \frac{1}{2} \left( J[D, \overline{D}]J + 2\Phi\overline{\Phi}' + \frac{2}{3}J^3 - 4J\Phi\overline{\Phi} \right) . \quad (19)$$

They coincide (up to rescalings of supercurrents and conventions on the algebra of  $N = 2$  spinor derivatives) with the explicit expressions for the first three hamiltonians of  $N = 4$  KdV hierarchy given in [5].

For completeness let us write here the corresponding flows of the  $N = 4$  KdV hierarchy. Besides the first flow ( $\dot{\Omega} = \Omega'$ ,  $\Omega \equiv \{J, \Phi, \overline{\Phi}\}$ ) we have

$$\frac{\partial}{\partial t_2} J = [D, \overline{D}] J' + 2JJ' - 2(\Phi\overline{\Phi})' , \quad \frac{\partial}{\partial t_2} \Phi = \Phi'' - 2D\overline{D}(J\Phi) . \quad (20)$$

---

<sup>1</sup>The precise correspondence with, e.g., ref. [8] is achieved by rescalings  $J \rightarrow -2J, \partial \rightarrow -\partial, \{ , \} \rightarrow \frac{1}{2}\{ , \}$ .

Our hamiltonians correspond to a particular choice ( $a = 4, b = 0$  in [5]), for the  $N = 4$  integrable hierarchy involving  $J, \Phi, \overline{\Phi}$ . It has been proven in [5] that there exist at most two  $N = 4$  integrable hierarchies characterized by different choices of the coefficients. However the two hierarchies so characterized are related to each other through a global transformation of the  $N = 4$  supersymmetry automorphism group  $SU(2)$ . This means that they actually define the same  $N = 4$  hierarchy. This point has been fully analyzed in [5], both with the use of harmonic superspace and in the  $N = 2$  formalism. We limit ourselves to pointing out that in the language of  $N = 2$  affine superfields the second choice of the coefficients would correspond to hamiltonians which do not obey the  $U(1)$  chargeless condition  $Q(\mathcal{H}) = 0$ . As follows from the consideration in ref. [5], they still should respect some hidden  $U(1)$  invariance. Their precise relation to the above hamiltonians with manifest  $U(1)$  invariance can be revealed by finding out how the automorphism group  $SU(2)$  is realized on  $H, \overline{H}, F, \overline{F}$  and then performing an appropriate  $SU(2)$  rotation of these hamiltonians.

It should be also stressed that we cannot exclude the possibility that other  $N = 4$  hierarchies, involving the superfields  $H, \overline{H}, F, \overline{F}$  but not expressible only through  $J, \Phi, \overline{\Phi}$ , could indeed exist. We leave the complete analysis of this problem for the future work.

We end this section with two comments.

The first one regards a hidden  $N = 4$  structure of the  $N = 2$   $sl(2) \widehat{\oplus} u(1)$  algebra. It turns out that the latter gives rise, besides the “small”  $N = 4$  SCA, to a wider  $N = 4$  SCA, namely to the “large”  $N = 4$  SCA in a specific realization. Indeed, one can define one more  $N = 4$  supermultiplet of composite supercurrents

$$\begin{aligned} \hat{J} &= H\overline{H} + F\overline{F}, \quad \hat{\Phi} = HF, \quad \hat{\overline{\Phi}} = \overline{F}H, \\ \delta\hat{J} &= \bar{\epsilon}\overline{D}\hat{\Phi} - \epsilon D\hat{\overline{\Phi}}, \quad \delta\hat{\Phi} = -\epsilon D\hat{J}, \quad \delta\hat{\overline{\Phi}} = -\bar{\epsilon}\overline{D}\hat{J}, \end{aligned} \quad (21)$$

which satisfy the PBs (18), but with vanishing central terms. In other words, they generate a kind of topological “small”  $N = 4$  SCA. Together with  $J, \Phi, \overline{\Phi}$  these additional supercurrents close on “large”  $N = 4$  SCA in a particular realization with 3 elementary bosonic  $sl(2)$  spin 1 affine currents and three composite ones constructed out of the elementary fermionic spin 1/2 currents. The presence of this hidden “large”  $N = 4$  SCA in  $N = 2$  affine  $sl(2) \widehat{\oplus} u(1)$  algebra has been earlier noticed, at the full quantum level, in [15]. This feature can be considered as an indication that the presented  $N = 4$  NLS-mKdV hierarchy bears a relation to a more general super KdV hierarchy (still to be constructed) associated with the whole “large”  $N = 4$  SCA.

The second comment is related to  $N = 2$  extension of the abelian affine algebra  $\oplus_{i=1}^4 \widehat{u(1)}_i$ , eqs. (9). As was noticed in Sect. 2, it also reveals a covariance under the rigid  $N = 4$  supersymmetry, this time realized by linear transformations (10) (combined with those of manifest  $N = 2$  supersymmetry). One may wonder whether it also gives rise to  $N = 4$  SCAs via some kind of Sugawara construction and has any relation to  $N = 4$  KdV hierarchy. It is easy to check that one cannot construct, out of the superfields  $H_\alpha, \overline{H}_\beta$ , any  $N = 4$  multiplet of composite currents which would include  $N = 2$  conformal stress-tensor with a Feigin-Fuks term (the latter is absolutely necessary for producing a central term in  $N = 2$  SCA and thus generating at least an  $N = 2$  KdV hierarchy). The only possibility is the  $N = 4$  multiplet

$$\hat{J} = H_1\overline{H}_1 + H_2\overline{H}_2, \quad \hat{\Phi} = H_1H_2, \quad \hat{\overline{\Phi}} = \overline{H}_2\overline{H}_1, \quad (22)$$



which, via PBs (9), generates a topological “small”  $N = 4$  SCA. So, possible  $N = 2$  hierarchies (even possessing rigid  $N = 4$  supersymmetry) constructed on the basis of  $N = 2 \oplus_{i=1}^4 \widehat{u(1)}_i$  algebra seem to have no any direct relation to  $N = 4$  KdV hierarchy. We have also explicitly checked that there exists no  $N = 4$  covariant system of evolution equations for  $H_\alpha, \overline{H}_\beta$  which would reduce to  $N = 2$  NLS system of refs. [11, 12] after putting one pair of these superfields equal to zero. Thus the minimal way to define an mKdV type hierarchy for  $N = 4$  KdV system (yielding also  $N = 2$  NLS hierarchy as a consistent reduction) is based on the use of non-abelian affine superalgebra  $N = 2 \widehat{sl(2) \oplus u(1)}$ . The nonlinearity of rigid  $N = 4$  transformations (7) is crucial for constructing the  $N = 4$  multiplet of supercurrents (14), (16) with a non-trivial  $N = 2$  SFF stress-tensor.

**5. Lax operator.** Due to the Sugawara-Feigin-Fuks construction which maps the affine superfields  $H, \overline{H}, F, \overline{F}$  onto the superfields generating the  $N = 4$  SCA there is no further need to prove the integrability of the  $N = 4$  NLS-mKdV hierarchy. We are indeed guaranteed by the integrability property of the  $N = 4$  KdV hierarchy that the corresponding affine hierarchy is integrable. The higher order hamiltonians in involution are in fact expressed through  $J, W, \overline{W}$  and therefore, via SFF, through  $H, \overline{H}, F, \overline{F}$ .

It is, however, important to realize that the Lax operator for the  $N = 4$  KdV [7] indeed furnishes the correct evolution equations for the affine superfields, once the basic superfields are re-expressed by the SFF construction. This statement has to be verified independently because, to our knowledge, no general relationship was as yet established between the  $N = 2$  affine PB structure and the Lax operators we are considering. So we made this checking explicitly.

The  $N = 4$  NLS-mKdV scalar Lax operator constructed according to the above prescription reads

$$L = D\overline{D} + D\overline{D}\partial^{-1} \left( J + \overline{\Phi}\partial^{-1}\Phi \right) \partial^{-1} D\overline{D}, \quad (23)$$

where  $J$  and  $\Phi, \overline{\Phi}$  are expressed by eqs. (14), (16). The flows are given by

$$\frac{\partial}{\partial t_k} L = -[L^k_{\geq 1}, L] \quad (24)$$

where the suffix  $\geq 1$  means taking the strictly differential part of the operator.

**6.  $N = 2$  reductions and bosonic cores.** In this section we discuss different  $N = 2$  supersymmetric integrable reductions of the  $N = 4$  NLS-mKdV hierarchy.

*i)* Let us firstly set  $F = \overline{F} = 0$ . This is a consistent reduction for all flows as all the time derivatives of  $F$  are zero in this limit. The hierarchy thus produced is expressed in terms of the chiral and antichiral superfields  $H, \overline{H}$  generating the  $u(1) \widehat{\oplus} u(1)$  subalgebra as the relevant second hamiltonian structure. It is the  $N = 2$  mKdV hierarchy associated with the  $a = 4$ ,  $N = 2$  KdV system. Indeed, upon this reduction,  $\Phi = \overline{\Phi} = 0$ ,  $J = (H\overline{H} + D\overline{H} + \overline{D}H)$ , and the closed set of the evolution equations for  $J$  corresponds just to the  $a = 4$ ,  $N = 2$  KdV. This reduction can be performed directly in the Lax representation (23), (24).

*ii)* Let us set  $H = \overline{H} = 0$ . Then

$$J = F\overline{F}, \quad \Phi = D\overline{F}, \quad \overline{\Phi} = \overline{D}F, \quad DF = \overline{D}\overline{F} = 0. \quad (25)$$

This is also a consistent reduction because both the left- and right-hand sides of the evolution equations for  $H = \overline{H} = 0$  vanish upon effecting it. It cannot be directly performed at the

algebraic level, we have to implement the Dirac's bracket formalism to deduce the relevant second hamiltonian structure algebra<sup>2</sup>. Nevertheless, it goes straightforwardly at the level of the equations and conserved hamiltonians and yields the  $N = 2$  NLS hierarchy of refs. [11, 12]. It is a simple exercise to check that the conserved quantities  $H_1 - H_4$  of  $N = 4$  NLS-mKdV (or  $N = 4$  KdV) hierarchy go over to the corresponding quantities of  $N = 2$  NLS one (with  $H = \overline{H} = 0$ ) upon substituting the expressions (25) for  $J$  and  $\Phi, \overline{\Phi}$ . The fact that the reduction (25) takes the Lax operator for  $N = 4$  KdV hierarchy into that for  $N = 2$  NLS has been earlier noticed in [6].

iii) There exists one more, rather unexpected reduction. It goes by imposing a sort of “mixed” constraint,  $\overline{H} = 0$  and  $F = 0$  (or equivalently  $H = 0$  and  $\overline{F} = 0$ ). The constraint  $F = 0$  is allowed due to the global charge conservation, but it turns out that also  $\overline{H} = 0$  is admitted by the equations. As in the previous case, these two constraints can be straightforwardly imposed at the level of the evolution equations while requiring a more subtle treating at the level of PBs. For completeness we give here the flows corresponding to this “mixed” reduced  $N = 2$  hierarchy. It would be interesting to analyze its possible relation to other known  $N = 2$  hierarchies. Besides the trivial first flow we have

$$\frac{\partial H}{\partial t_2} = H'' + 2\overline{D}HH', \quad \frac{\partial \overline{F}}{\partial t_2} = \overline{F}'' + 2\overline{D}H\overline{F}', \quad (26)$$

and

$$\begin{aligned} \frac{\partial H}{\partial t_3} &= -H''' - 3H''\overline{D}H - 3H'\overline{D}H\overline{D}H - 3\overline{D}H'H' \\ \frac{\partial \overline{F}}{\partial t_3} &= -\overline{F}''' - 3\overline{F}''\overline{D}H - 3\overline{F}'\overline{D}H\overline{D}H - 3\overline{F}'\overline{D}H'. \end{aligned} \quad (27)$$

Thus in both cases we get two non-linear equations, one involving the superfield  $H$  alone and the other describing an evolution of an anti-chiral superfield  $\overline{F}$  in the background of  $H$ .

Finally, let us present the bosonic limit of the second flow equations of our  $N = 4$  mKdV-NLS hierarchy. We define the bosonic components of the superfields  $H, \overline{H}, F, \overline{F}$  as follows

$$[\overline{D}F] = \phi, [D\overline{F}] = \overline{\phi}, [\overline{D}H] = h, [D\overline{H}] = \overline{h}. \quad (28)$$

With such a definition the bosonic core of our system (13) reads

$$\begin{aligned} \frac{\partial \phi}{\partial t_2} &= -\phi'' + 2h\phi' + 2h'\phi + 2\overline{h}\phi' + 2\phi\phi\overline{\phi} + 2h\overline{h}\phi, \\ \frac{\partial h}{\partial t_2} &= h'' + 2hh' + 2h\overline{h}' + 2h'\overline{h}. \end{aligned} \quad (29)$$

**7. One more  $N = 2$  hierarchy with the  $sl(2) \widehat{\oplus} u(1)$  structure.** In [8] it has been proven that “small”  $N = 4$  SCA provides the PB structure for one more hierarchy of integrable equations whose hamiltonians respect only  $N = 2$  supersymmetry. It is an extension of the  $a = -2$ ,  $N = 2$  KdV hierarchy and is related via a non-polynomial Miura transformation to

---

<sup>2</sup>This non-local structure is explicitly quoted in [16].

the  $\alpha = -2$ ,  $N = 2$  super Boussinesq hierarchy (and hence to  $N = 2$   $\mathcal{W}_3$  superalgebra which is the second PB structure for all three known  $N = 2$  Boussinesq hierarchies). Its characteristic feature is that all the even-dimensional hamiltonians are vanishing in the  $a = -2$ ,  $N = 2$  KdV limit  $\Phi = \overline{\Phi} = 0$ . First non-trivial hamiltonian is given by the following integral over  $N = 2$  superspace

$$H_2 = \int dZ \Phi \overline{\Phi} = \int dZ D \overline{F} \overline{D} F. \quad (30)$$

It is clear that the existence of this hierarchy implies, via the  $N = 4$  SFF construction (14), (16), the existence of the underlying mKdV type heirarchy of evolution equations for the  $N = 2$  affine superfields  $H, \overline{H}, F, \overline{F}$ . Indeed, substituting into the relevant hamiltonians the expressions (14), (16) for  $J$  and  $\Phi, \overline{\Phi}$  one easily deduces the equations for  $H, \overline{H}, F, \overline{F}$  using the PBs algebra (1) - (3). We limit ourseleves by explicitly presenting the second flow equations associated with  $H_2$ , eq. (30)

$$\begin{aligned} \frac{\partial F}{\partial t_2} &= -F'' + F' F \overline{F} - H' \overline{D} F + D \overline{H} F' + \overline{D} H' F + \overline{D} H F' - \overline{D} H D \overline{H} F \\ &\quad + H D \overline{H} D F - H \overline{D} H \overline{H} F + H \overline{H} F' + H F \overline{D} F \overline{F}, \\ \frac{\partial H}{\partial t_2} &= -F D \overline{F}' - F' D \overline{F} - H F \overline{F}' - H \overline{D} F D \overline{F} + \overline{D} H F D \overline{F} \\ &\quad - H \overline{H} F D \overline{F} - H D \overline{H} F \overline{F}. \end{aligned} \quad (31)$$

To all consistent reductions of the “quasi”  $N = 4$  KdV hierarchy listed in [8] there correspond the appropriate reductions of this underlying mKdV type hierarchy. In particular, the reduction  $F = \overline{F} = 0$  is consistent, and it takes the above hierarchy into the  $N = 2$  mKdV one corresponding to the  $a = -2$ ,  $N = 2$  KdV hierarchy. Upon this reduction all the even-dimension hamiltonians vanish and the corresponding flows become trivial. Note that no reduction  $H = \overline{H} = 0$  exists in the present case, which reflects the fact that no analog of  $N = 2$  NLS hierarchy can be defined for the  $a = -2$ ,  $N = 2$  KdV.

Like in the previous case, scalar Lax representation for the above hierarchy can be obtained by substituting (14), (16) into the “quasi”  $N = 4$  KdV heirarchy Lax operator given in [8, 7]

$$\begin{aligned} L &= D \left( \partial + H \overline{H} + F \overline{F} + (\overline{D} H) + (D \overline{H}) - (D \overline{F}) \partial^{-1} (\overline{D} F) \right) \overline{D}, \\ \frac{\partial}{\partial t_k} L &= - \left[ L_{\geq 1}^{\frac{k}{2}}, L \right]. \end{aligned} \quad (32)$$

**8. Conclusions.** In this paper we have succeeded for the first time in establishing a link between  $N = 2$  affine Lie algebras on one hand and large  $N$  supersymmetric extended hierarchies on the other. We did so by showing that the  $N = 2$  affine  $\widehat{sl(2)} \oplus u(1)$  algebra reveals a global  $N = 4$  supersymmetry and yields a PB structure for a globally  $N = 4$  supersymmetric integrable hierarchy which generalizes both  $N = 2$  NLS and  $N = 2$  mKdV ones. We proved that an appropriate Sugawara-Feigin-Fuks construction leads to the “small”  $N = 4$  SCA and to the related  $N = 4$  KdV. As a by-product of our method some other constructions have been worked out. In particular we have furnished a new global  $N = 2$  hierarchy based on the same affine algebra. Lax operators have been given for both these hierarchies.

It is clear that this work opens a way to further developments. Just as in the purely bosonic case, where the constructions based on affine Lie algebras have paved the way towards an understanding and a classification of all hierarchies, a similar situation can now be faced for the  $N = 4$  hierarchies. The strategy is clear, it should consist in selecting affine algebras with quaternionic structure and analyzing the properties of their affinizations, sequences of invariant hamiltonians in involutions, Sugawara constructions, etc. The next simplest case of interest involves the  $N = 2$  superaffine  $\widehat{sl(3)}$  algebra, which one naturally suspects to lead to  $N = 4$  generalizations of (some of) the  $N = 2$  Boussinesq hierarchies and to a sort of  $N = 4$  extensions of  $W_3$  algebra.

All things could probably be made much more simple and transparent in a manifestly  $N = 4$  supersymmetric formalism, based, e.g., on the harmonic superspace approach.

Finally, we note that in a recent paper [17]  $N = 4$  KdV hierarchy was mapped on the so called  $(1, 1)$  GNLS hierarchy [18] defined on a pair of chiral and anti-chiral fermionic and bosonic superfields. It is unclear to us which algebraic structure underlies this non-polynomial map and whether it bears any relation to the construction presented here.

## Acknowledgments

E.A. and S.K. acknowledge a partial support from the RFBR-DFG project No 9602-00180. Their work was also partly supported by RFBR under the project No 96-02-17634, INTAS under the project INTAS-94-2317 and by a grant of the Dutch NWO organization. F.T. would like to thank the Director of the Bogoliubov Laboratory of Theoretical Physics (JINR, Dubna) Academician D.V. Shirkov for the hospitality at BLTP during the course of this work. S.K. would like to thank D. Lüster for the hospitality at Humboldt University (Berlin) on the early stage of this study.

## References

- [1] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, Phys. Reports **254** (1995) 1
- [2] N. Seiberg and E. Witten, Nucl. Phys. **B 426** (1994) 19; A. Gorski, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. **B 355** (1995) 466
- [3] Yu. I. Manin and A.O. Radul, Commun. Math. Phys. **98** (1985) 65; P.P. Kulish, Lett. Math. Phys **10** (1985) 87
- [4] F. Delduc and E. Ivanov, Phys. Lett. **B 309** (1993) 312
- [5] F. Delduc, E. Ivanov and S. Krivonos, J. Math. Phys. **37** (1996) 1356
- [6] E. Ivanov and S. Krivonos, “New integrable extensions of  $N = 2$  KdV and Boussinesq hierarchies”, Preprint JINR E2-96-344, hep-th/9609191
- [7] F. Delduc and L. Gallot, “ $N = 2$  KP and KdV hierarchies in extended superspace”, Preprint ENSLAPP-L-617/96, solv-int/9609008

- [8] F. Delduc, L. Gallot and E. Ivanov, “New super KdV system with the  $N = 4$  SCA as the Hamiltonian structure”, Preprint ENSLAPP-L-623/96, JINR E2-96-394, hep-th/9611033
- [9] E. Ivanov and S. Krivonos, Phys. Lett. **B 291** (1992) 63; **B 301** (1993) 454 (E)
- [10] Ph. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, Phys. Lett. **B 206** (1988) 71
- [11] S. Krivonos and A. Sorin, Phys. Lett. **B 357** (1995) 94
- [12] S. Krivonos, A. Sorin and F. Toppan, Phys. Lett. **A 206** (1995) 146
- [13] C.M. Hull and B. Spence, Phys. Lett. **B 241** (1989) 357
- [14] C. Ahn, E. Ivanov and A. Sorin, Commun. Math. Phys. **183** (1997) 205
- [15] M. Rocek, C. Ahn, K. Schoutens and A. Sevrin, “Superspace WZW Models and Black Holes”, in Workshop on Superstrings and Related Topics, Trieste, Aug. 1991., IASSNS-HEP-91/69, ITP-SB-91-49, LBL-31325, UCB-PTH-91/50
- [16] V. Derjagin, A. Leznov and A. Sorin, “ $N = 2$  superintegrable f-Toda mapping and super-NLS hierarchy in  $(1|2)$  superspace”, Preprint JINR E2-96-410, hep-th/9611108
- [17] A. Sorin, “The discrete symmetries of the  $N = 2$  supersymmetric GNLS hierarchies”, solv-int/9701020
- [18] L. Bonora, S. Krivonos and A. Sorin, Nucl. Phys. **B 477** (1996) 835